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1986 J. Phys. A: Math. Gen. 19 1761

(http://iopscience.iop.org/0305-4470/19/10/017)

J. Phys. A: Math. Gen. 19 (1986) 1761-1773. Printed in Great Britain

Coupling coefficients of $SU_n \supset SO_n$ for O_n-scalar microscopic theory of collective states

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Received 13 June 1985

Abstract. Boson realisations of all O_n-scalar states which span a subspace of the Hilbert space for the collective degrees of freedom of the nucleus and are labelled by irreducible representations of the chain of subgroups $U_3 \supset U_2 \supset U_1$ of the complementary group Sp(6, R) are given. Two general expressions for isoscalar factors (reduced Wigner coefficients) of $SU_n \supset SO_n$ needed for constructing O_n-scalar basis functions depending on the microscopic collective variables in the oscillator basis are derived. These isoscalar factors couple three symmetric representations of $SU_n(U_n)$ $[p_1 0] \times [p_2 0] \times [p_3 0]$ to the representation $[h_1 h_2 h_3]$ of SU_n subduced to the $SO_n(O_n)$-scalar representation and, more generally, to the SO_n irreducible representation $[fff]$, which appears in the case of the closed shells.

1. Introduction

In the early 1970s Dzublik (1971, see Dzublik *et al* 1972) and Zickendraht (1971) introduced the microscopic collective and internal variables of the nucleus. The general features of the wavefunction depending on the collective and internal variables and, in particular, their group-theoretical characteristics were described by Vanagas and Kalinauskas (1973). Vanagas (1976, 1977) demonstrated that the microscopic consideration of collective effects is associated with the restriction of the states to a definite irreducible representation (irrep) of the O_n group ($A = n + 1$ is the number of nucleons).

An explicit construction of $U_{3n} \supset U_3 \times U_n \supset O_n$ basis functions depending on the microscopic collective variables was proposed by Vanagas and Kalinauskas (1974, see Vanagas 1977, 1980, Vanagas and Katkevičius 1983, Katkevičius and Vanagas 1983). For this purpose the special isoscalar factors (reduced Wigner coefficients or, briefly, isofactors) of the chain $SU_n \supset SO_n$ are needed which couple the states of the three symmetric irreps to the states of the three-rowed irrep of SU_n subduced to the corresponding irrep of SO_n.

As was shown by Vanagas (1977, 1980), the O_n-scalar subspace of Hilbert space for the microscopic collective degrees of freedom is isomorphic to the Hilbert space for the phenomenological collective model, based on the chain $U_6 \supset U_3$ introduced by Vanagas *et al* (1975a, b). Later this question was also considered by Vanagas (1981), Chacón *et al* (1981) and Deenen and Quesne (1982).

In another aspect the phenomenological chain $U_6 \supset U_3$ was used for the quadrupole phonon model (Janssen *et al* 1974, Jolos and Janssen 1977) and the interacting boson model (Arima and Iachello 1975, 1978). The relationships between the different approaches of using $U_6 \supset U_3$ for consideration of the collective phenomena are discussed by Vanagas (1982).

For explicit construction of the corresponding microscopic basis functions a particular class of the above-mentioned isofactors of $SU_n \supset SO_n$ is needed.

Recently Hecht and Suzuki (1983) used some generating functions in Bargmann space, which allowed them to obtain the expressions for the special $SU_3 \supset SO_3$ isofactors for coupling $(p_1 0)\times(p_2 0)$ to $(\lambda\mu)$ with the resulting irrep of SU_3 subduced to a scalar of SO_3. These methods were generalised for the $SU_n \supset SO_n$ case by Ališauskas (1984) for arbitrary two-rowed resulting irreps of SU_n and SO_n.

The main result of this paper is two expressions of $SU_n \supset SO_n$ isofactors for coupling $[p_1 0]\times[p_2 0]\times[p_3 0]$ to the irrep $[h_1 h_2 h_3]_n$ of SU_n, subduced to the scalar irrep of SO_n (or, more generally, to the irrep $[fff]$).

In order to obtain the generating function for such isofactors, the realisations of scalars of O_n in the irreducible spaces of U_n are found, which span the basis for the irrep $\langle n/2, n/2, n/2\rangle$ of $Sp(6, R)$ labelled by the chain of subgroups $U_3 \supset U_2 \supset U_1$. The first realisation leads to the simplest explicit expansion of the arbitrary normalised $U_3 \supset U_2 \supset U_1$ states belonging to the irrep of $Sp(6, R)$ under consideration in terms of the boson creation operators. It coincides with the result by Deenen and Quesne (1982), including the normalisation factors found by Castanos *et al* (1984) for the highest weight states of U_3. This realisation may be usefully applicable for constructing $SU_n \supset SO_n$ states in the case of more general irreps of SO_n according to Deenen and Quesne (1983). Besides, it gives (with a slightly different normalisation factor) a realisation of the phenomenological $U_6 \supset U_3 \supset U_2 \supset U_1$ states.

However, the second realisation of O_n scalars in terms of the elementary scalars of O_n and the generators of U_3 was more convenient in the role of generating function which allowed the expansion of the general isofactors under consideration in terms of a particular class of them. The latter, which couple the states of three symmetric irreps to the states of a two-rowed irrep of U_n, were found similarly to the isofactors of Elliott's states of $SU_3 \supset SO_3$ by Engeland (1965, cf Vergados 1968, Asherova and Smirnov 1970) with the help of the resubduction technique and the special isofactors of $U_n \supset U_2 \dotplus U^*_{n-2}$, $U_2 \supset SO_2$ and $SO_n \supset SO_2 \dotplus SO_{n-2}$†.

An alternative approach (expansion of the direct product states in terms of the coupled states) allowed us to obtain a different expression for the special isofactors under consideration. In order to find the general isofactors of this type, this result needs to be used together with the expressions for isofactors for coupling $[p_1 0]\times[p_2 0]$ to $[h'_1 h'_2]$ given by Ališauskas (1984)‡. In certain cases the first or second approach gives a more convenient result.

A related problem is consideration of the closed shell case when the irrep of O_n is characterised by three equal integers $[fff]$ (Vasilevsky *et al* 1980, Castanos *et al* 1984). The relation between the isofactors of the groups of different ranks (see Ališauskas 1983, 1984) allows us to find corresponding isofactors of $SU_n \supset SO_n$ in the case of the resulting irrep $[fff]$ of SO_n.

In the case of the resulting scalar irrep of SO_n all the Clebsch-Gordan (Wigner) coefficients for the chain $SU_n \supset SO_n \supset SO_{n-1} \supset SO_{n-2} \ldots$ may be factorised as a product of the isofactors under consideration and the special isofactors of $SO_k \supset SO_{k-1}$ ($k \leq n$,

† The sign $\dotplus$ is used here to mean the direct sum of Lie algebras, or the group matrices.

‡ In addition to corrections given in the corrigendum (1985) of Ališauskas (1984) it is necessary to change the corresponding factors to $(l_1+l_2-L_1+L_2+n-4-2x+2y)!!$ in (3.4), to $[\frac{1}{2}(\nu-l_2)-x+y]!$ in (7.1), to $[\frac{1}{2}(\mu-l_2-l'_{20}+\Delta+\delta)]!$ in (A3.5) and to $(\lambda+\mu-\bar{l}_1-1+2z)!!$ in (A3.9). The sign before δ' in the phase factor of (A2.3) needs to be changed to the opposite one. In the RHS of (5.2) the factor 2^{-n+3} is omitted. In explanations of (5.4) and (6.7) $\nu-L_2-\Delta_0$ and $\lambda+\nu-L_2-\delta_0$ are even.

see Norvaišas and Ališauskas (1974a) and (35*b*) of Ališauskas (1983)†). However, the special isofactors found by Norvaišas and Ališauskas (1974a, b) are not sufficient for the factorisation of the Clebsch-Gordan coefficients in the case of more general resulting irreps of SO_n. Some new and more convenient expressions for the special isofactors of $SO_n \supset SO_{n-1}$ connected with the above-mentioned problem of Vanagas (1977) will be published in a future paper.

2. Polynomial O_n-scalar states with $U_3 \supset U_2 \supset U_1$ labels

The polynomial O_n-scalar states in Bargmann space depending on $3n$ boson creation operators η_{is} $(i=1,2,3;\ s=1,2,\ldots,n)$ belong to a single irrep $\langle n/2, n/2, n/2\rangle$ of $Sp(6,R)$. Let us denote the corresponding states of irreps for the chains of the complementary groups $Sp(6,R) \supset U_3 \supset U_2 \supset U_1$ and $(U_n \times U_n) \times U_n \supset U_n \times U_n \supset U_n \supset SO_n$ by

$$\left|\begin{matrix}[h_1h_2h_3]_n \\ \alpha, IM; 0\end{matrix}\right\rangle \tag{2.1}$$

where h_1, h_2, h_3 are even integers and together with other parameters form the Gelfand-Zetlin tableau of U_3

$$\left|\begin{matrix} h_1 & & h_2 & & h_3 \\ & \alpha+I & & \alpha-I & \\ & & \alpha+M & & \end{matrix}\right\rangle. \tag{2.2}$$

Deenen and Quesne (1982) (cf Vasilevsky *et al* 1980) give the highest weight states (HWS) in terms of the elementary scalars of O_n,

$$W_{ij} = \sum_{s=1}^{n} \eta_{is}\eta_{js} \qquad (i,j=1,2,3), \tag{2.3}$$

which are non-compact generators of $Sp(6,R)$. The normalisation factor is found by Castanos *et al* (1984). However, the suggested application of the U_3 lowering operators as the next step is not the simplest way to construct an arbitrary state of U_3. It is expedient to construct at first the SU_2-scalar state and later on the states for $M=I$ corresponding to the definite irrep of U_3 by

$$\begin{aligned}\left|\begin{matrix}[h_1h_2h_3]_n \\ \alpha, II; 0\end{matrix}\right\rangle =& \left(\frac{(n-2)!(n-4)!!(h_1-h_2+1)(h_2-h_3+1)(h_1-h_3+2)}{(h_1+n-2)!!(h_2+n-3)!!(h_3+n-4)!!(h_1+2)!!(h_2+1)!!}\right.\\ &\times\left.\frac{(h_1-\alpha-I)!(\alpha-I-h_3)!(2I+1)!}{h_3!!(h_2-\alpha+I)!(h_1-\alpha+I+1)!(\alpha+I-h_2)!(\alpha+I-h_3+1)!}\right)^{1/2}\\ &\times\sum_z \frac{(-1)^z[\frac{1}{2}(h_1-h_3)-z]!}{z![\frac{1}{2}(h_2-h_3)-z]![\frac{1}{2}(h_1-h_2)-z]!} E_{32}^{h_2-\alpha+I} E_{13}^{\alpha+I-h_2}\\ &\times W_{123,123}^{h_3/2+z} W_{12,12}^{(h_2-h_3)/2-z} W_{33}^{(h_1-h_2)/2-z}|0\rangle\end{aligned} \tag{2.4}$$

† Some phase changes and corrections of the first paper are discussed by Ališauskas and Norvaišas (1980). The factors 2 are omitted before z_i in the last row of (35*a*) of Ališauskas (1983), as well as the factor $(2l_3+n-2)^{1/2}$ in the RHS of (38).

where $W_{123,123}$ and $W_{12,12}$ are the corresponding minors of $\det|W|$ (see Deenen and Quesne 1982, 1983, Vasilevsky *et al* 1980),

$$E_{ij} = \sum_{s=1}^{n} \eta_{is}\bar{\eta}_{js} = \sum_{s=1}^{n} \eta_{is}\frac{\partial}{\partial \eta_{js}} \tag{2.5}$$

are generators of U_3 and $\bar{\eta}_{js}$ are the boson annihilation operators. Here the operators E_{13}^a and E_{32}^b transform only the last and the last but one factors in the RHS of (2.4) respectively. For example,

$$E_{13}^a W_{33}^c = \sum_t \frac{a!c!2^{a-2t}}{t!(c-a+t)!(a-2t)!} W_{11}^t W_{13}^{a-2t} W_{33}^{c-a+t}, \tag{2.6a}$$

$$E_{32}^b W_{12,12}^d = \sum_s \frac{b!d!2^{b-2s}}{s!(d-b+s)!(b-2s)!} W_{13,13}^s W_{12,13}^{b-2s} W_{12,12}^{d-b+s}. \tag{2.6b}$$

Let us replace the elementary O_n-scalars W_{ij} in the RHS of (2.4) by $(1+\delta_{ij})^{1/2}\xi_{ij}$, where the operators ξ_{ij} are the boson creation operators of the phenomenological $U_6 \supset U_3$ states (Deenen and Quesne 1982), and omit the factors depending on n. In this way the normalised states for the chain $U_6 \supset U_3 \supset U_2$ are obtained.

In order to prove that the function (2.4) is the state of the definite irrep of U_3, the factors obtained from W_{33}^c and $W_{12,12}^d$ may be expressed as some states for $U_6 \supset U_3 \supset U_2$ (see Vanagas *et al* 1980, Quesne 1981). The use of the special isofactors of $U_6 \supset U_3$ (Norvaišas 1983), $U_3 \supset U_2$ (Ališauskas and Jucys 1967a) and the elementary reduced matrix elements allowed us to expand the separate summands of the new variant of (2.4) in terms of the irreducible tensors of U_3. After summation over z the state of a single irrep of U_3 remains in the RHS of (2.4).

The inverse mapping allows us to show (2.4) to be valid for the special $U_n \supset O_n$ states† including normalisation which may be checked by comparison with HWS found by Castanos *et al* (1984). The states with $M < I$ may be obtained by acting on (2.4) with the operator

$$\left(\frac{(I+M)!}{(2I)!(I-M)!}\right)^{1/2} E_{21}^{I-M}. \tag{2.7}$$

The arbitrary state of $U_3 \supset U_2 \supset U_1$ also may be generated from the HWS by means of (2.7) and the following relation:

$$\begin{aligned} \left| \begin{matrix} [h_1h_2h_3]_n \\ \alpha, I, M; 0 \end{matrix} \right\rangle &= \left\langle \begin{matrix} [h_1h_2h_3] \\ [\alpha+I, \alpha-I] \end{matrix} \right\| E_{32}^{2j_1} \left\| \begin{matrix} [h_1h_2h_3] \\ [h_1h_2] \end{matrix} \right\rangle^{-1} \sum_{M_1 m_1} \begin{bmatrix} \frac{1}{2}(h_1-h_2) & j_1 & I \\ M_1 & m_1 & M \end{bmatrix} \\ &\times (-1)^{j_1-m_1} \left(\frac{(2j_1)!}{(j_1+m_1)!(j_1-m_1)!}\right)^{1/2} E_{32}^{j_1+m_1} E_{31}^{j_1-m_1} \\ &\times \left| \begin{matrix} [h_1h_2h_3]_n \\ \frac{1}{2}(h_1+h_2), \frac{1}{2}(h_1-h_2), M_1; 0 \end{matrix} \right\rangle \end{aligned} \tag{2.8}$$

where $j_1 = \frac{1}{2}(h_1+h_2) - \alpha$. On the RHS both the reduced matrix element (related to one given by Ališauskas (1969, 1983), Ališauskas *et al* (1972) and Chacón *et al* (1972)) and the Clebsch-Gordan coefficient of SU_2 appeared.

† The RHS of (2.4) being the state of the definite irrep of U_n may be proved immediately in the frame of $U_n \supset SO_n$ with the help of the special case of the isofactor (3.9) and some methods used in § 4, together with the normalisation coefficients of Castanos *et al* (1984).

The resubduction matrices (transformation brackets) between the chains $SU_3 \supset U_2 \supset U_1$ and $SU_3 \supset SO_3$ (see Moshinsky 1962, Moshinsky and Devi 1969, Asherova and Smirnov 1970, Moshinsky *et al* 1975, Ališauskas 1978) allow us to find O_n-scalar states with $U_3 \supset SO_3 \supset SO_2$ labels as well.

However, neither equation (2.4) nor (2.8) together with the result of Deenen and Quesne (1982) gives sufficiently convenient generating functions for the required isofactors of $U_n \supset SO_n$. In this respect the following relation seems to be the best one:

$$\left|\begin{matrix}[h_1h_2h_3]_n \\ \alpha, I, M; 0\end{matrix}\right\rangle = \left(\frac{(h_1-h_2+1)(h_2-h_3+1)(h_1-h_3+2)(h_1+2)!!(h_2+1)!!h_3!!}{(h_1+n-2)!!(h_2+n-3)!!(h_3+n-4)!!}\right)^{1/2}$$
$$\times\frac{1}{S_{3,2}(h_1h_2h_3;\ \alpha+I,\ \alpha-I)}\sum_{\substack{I_0M_0\\ j,m}}\frac{(-1)^{(h_1-h_2)/2+I-m}}{(h_1+h_2+h_3-2\alpha-2j)!!}$$
$$\times\left(\frac{(2I_0+1)(\alpha+j+I_0+n-2)!!(\alpha+j-I_0+n-3)!!}{(\alpha+j-I_0)!!(\alpha+j+I_0+1)!!(j+m)!(j-m)!}\right)^{1/2}$$
$$\times\frac{(h_1-\alpha-j+I_0-1)!!\nabla(j, I_0, I)}{(\alpha-h_1+I_0+j)!!(\alpha-h_2-I_0+j)!!(\alpha-h_2+I_0+j+1)!!}$$
$$\times[(\alpha-h_3-I_0+j+1)!!(\alpha-h_3+I_0+j+2)!!]^{-1}\begin{bmatrix} I_0 & j & I \\ M_0 & m & M\end{bmatrix}$$
$$\times W_{33}^{(h_1+h_2+h_3)/2-\alpha-j}E_{32}^{j+m}E_{31}^{j-m}\left|\begin{matrix}[\alpha+j+I_0,\ \alpha+j-I_0,\ 0]_n \\ \alpha+j,\ I_0,\ M_0;\ 0\end{matrix}\right\rangle. \tag{2.9}$$

Here

$$S_{3,2}(h_1h_2h_3;\ m_{12}m_{22}) = \left(\frac{(h_1-m_{12})!(h_2-m_{22})!(h_1-m_{22}+1)!}{(m_{12}-h_2)!(m_{22}-h_3)!(m_{12}-h_3+1)!}\right)^{1/2}, \tag{2.10}$$

$$\nabla(abc) = \left(\frac{(a+b-c)!(a-b+c)!(a+b+c+1)!}{(b+c-a)!}\right)^{1/2}, \tag{2.11}$$

the summation parameter I_0 is an integer and $\alpha+j\pm I_0$ are even integers. In the RHS of (2.9) the state of the two-rowed irrep of U_n appeared, which depends only on W_{11}, W_{12} and W_{22} and has considerably simpler structure than the state of the three-rowed irrep.

It is evident that the state (2.9) corresponds to the irrep I of SU_2 and has definite weight components. One needs to act with the operator E_{23} in order to check that the RHS of (2.9) is the HWS for $\alpha=\frac{1}{2}(h_1+h_2)$, $M=I=\frac{1}{2}(h_1-h_2)$ (cf Moshinsky 1962). The SU_2-tensorial properties of the operator E_{23}, the Wigner–Eckart theorem and other techniques of the angular momentum theory (see Jucys and Bandzaitis 1977, Biedenharn and Louck 1981) are to be used for this purpose. The relation (2.8) and the SU_2 recoupling technique allowed us to induce (2.9) for arbitrary values of the parameters.

The normalisation of the state (2.9) may be checked by comparison of the coefficients before the term

$$W_{11}^{h_1/2}W_{22}^{(h_2-h_3)/2}W_{23}^{h_3}$$

of the HWS in the forms (2.9) and (2.4). The corresponding coefficient in (2.9) appeared to contain a triple sum. The sums over two parameters were taken as elementary (see Jucys and Bandzaitis 1977, § 14, or Slater 1966, § 1.7). The last sum corresponds to the well posed ${}_5F_4(1)$ series and may be summed with the help of Dougall's theorem (see Slater 1966, § 2.3.4).

3. The first expression of the special isofactors for coupling $[p_1 0]\times[p_2 0]\times[p_3 0]$ to $[h_1 h_2 h_3]$

The use of the generating function (2.9) allowed us to write the following expression for the special isofactors of $U_n \supset SO_n$:

$$\sum_{\omega}\begin{bmatrix}[p_1 0] & [p_2 0] & [\alpha+I, \alpha-I]_n\\ l_1 & l_2 & \omega l_3\end{bmatrix}\begin{bmatrix}[\alpha+I, \alpha-I] & [p_3 0] & [h_1 h_2 h_3]_n\\ \omega l_3 & l_3 & 0\end{bmatrix}$$

$$=\left(\frac{(h_1-h_2+1)(h_2-h_3+1)(h_1-h_3+2)(h_1+2)!!(h_2+1)!!h_3!!}{(n-2)(n-3)!!(h_1+n-2)!!(h_2+n-3)!!(h_3+n-4)!!(\alpha-I)!(\alpha+I+1)!}\right)^{1/2}$$

$$\times\frac{W_n(p_3 l_3)}{S_{3,2}[h_1 h_2 h_3;\ \alpha+I,\ \alpha-I]}\sum_{I_0,j}\frac{(-1)^{(h_1+h_2-\alpha-I_0-j)/2+\delta_{n,3}I_0}}{2^{p_3/2-j}(\frac{1}{2}p_3-j)!}$$

$$\times\frac{(\alpha+j+I_0+n-2)!!(\alpha+j-I_0+n-3)!!(h_1-\alpha-j+I_0-1)!!}{(\alpha-h_1+I_0+j)!!(\alpha-h_2-I_0+j)!!(\alpha-h_2+I_0+j+1)!!}$$

$$\times\frac{\nabla(jI_0I)(2I_0+1)^{1/2}}{(\alpha-h_3+I_0+j+2)!!(\alpha-h_3-I_0+j+1)!!\,W_n(2j, l_3)}$$

$$\times\sum_{m_1 m_2}V_n(p_1, l_1, m_1)V_n(p_2, l_2, m_2)V_n(2j, l_3, m_1+m_2)$$

$$\times\begin{bmatrix}\frac{1}{2}p_1 & \frac{1}{2}p_2 & I\\ \frac{1}{2}m_1 & \frac{1}{2}m_2 & \frac{1}{2}(m_1+m_2)\end{bmatrix}\begin{bmatrix}I & j & I_0\\ \frac{1}{2}(m_1+m_2) & -\frac{1}{2}(m_1+m_2) & 0\end{bmatrix}$$

$$\times\left(\begin{array}{c|ccc|c}SO_n & l_1 & l_2 & l_3 & 0\\ SO_2\dotplus SO_{n-2} & m_1 0 & m_2 0 & -m_1-m_2, 0 & 0\ \ 0\end{array}\right). \tag{3.1}$$

Here I_0 is integer, $\alpha+j\pm I_0$ are even integers and ω is a multiplicity label.

On the RHS the Clebsch-Gordan coefficients of SU_2 are denoted by square brackets, and $V_n(p, l, m)$ are the special resubduction coefficients between the states of the symmetric irreps of U_n restricted to $U_n \supset (U_2 \dotplus U_{n-2}) \supset SO_2 \dotplus SO_{n-2}$ and $U_n \supset SO_n \supset SO_2 + SO_{n-2}$. They can be expressed as

$$V_n(p, l, m)\equiv\left\langle\begin{array}{c|c}[p]_n & [p]_n\\ [p]_2[0]_{n-2} & l\\ m\ \ 0 & m\ \ 0\end{array}\right\rangle$$

$$=(-1)^{\delta_{n,3}(p-l)/2}2^{(p-l)/2}[u_n(l, m)/W_n(p, l)]$$

$$\times\{(2l+n-2)[\tfrac{1}{2}(p-m)]![\tfrac{1}{2}(p+m)]!/(n-4)!!\}^{1/2} \tag{3.2}$$

(see Ališauskas and Vanagas 1972, Ališauskas 1983). The phase for $n=3$ is correlated with Moshinsky *et al* (1975) and for $n=4$ with one of the Clebsch-Gordan coefficients of SU_2. Here

$$u_n(l, m)=\left(\frac{(l-m+n-4)!!(l+m+n-4)!!}{[\frac{1}{2}(l-m)]![\frac{1}{2}(l+m)]!}\right)^{1/2}, \tag{3.3}$$

$$W_n(p, l)=[(p-l)!!(p+l+n-2)!!]^{1/2}. \tag{3.4}$$

The special Wigner coefficients of $SO_n \supset SO_2 \dotplus SO_{n-2}$ (for coupling three irreps to a scalar) are given by

$$\begin{pmatrix} SO_n & l_1 & l_2 & l_3 & 0 \\ SO_2 \dotplus SO_{n-2} & m_1 0 & m_2 0 & m_3 0 & 0\ 0 \end{pmatrix}$$
$$= \delta_{m_1+m_2,-m_3}(n-4)!![2^{n-3}(n-2)!]^{1/2}\nabla_{n[0,1,2,3]}(l_1 l_2; l_3 0)$$
$$\times \frac{u_n(l_3, m_3)}{u_n(l_1, m_1)u_n(l_2, m_2)} \sum_{y,z} \frac{(-1)^{y+z}(2l_1+n-4-2y)!!}{y!z![\frac{1}{2}(l_1+l_2-l_3)-y-z]!}$$
$$\times \frac{(2l_2+n-4-2z)!![\frac{1}{2}(l_1+l_2+m_3)-y-z]!}{[\frac{1}{2}(l_1-m_1)-y]![\frac{1}{2}(l_1+m_1)-y]![\frac{1}{2}(l_2-m_2)-z]![\frac{1}{2}(l_2+m_2)-z]!}$$
$$\times \frac{[\frac{1}{2}(l_1+l_2-m_3)-y-z]!}{(l_1+l_2+l_3+n-2-2y-2z)!!} \tag{3.5}$$

where

$$\nabla_{n[0,1,2,3]}(l_1 l_2; l_3 0)$$
$$= \left(\frac{(l_1+l_2-l_3)!!(l_3-l_1+l_2)!!}{(l_1+l_2-l_3+n-4)!!(l_3-l_1+l_2+n-4)!!}\right.$$
$$\left.\times \frac{(l_1-l_2+l_3)!!(l_1+l_2+l_3+n-2)!!}{(l_1-l_2+l_3+n-4)!!(l_1+l_2+l_3+2n-6)!!}\right)^{1/2}. \tag{3.6}$$

The coefficient (3.5) does not change its value after permutations of the parameters $l_1, m_1; l_2, m_2; l_3, m_3$. The general expression for isofactors of such type was found by Norvaišas and註Ališauskas (1974a) (see second footnote on p 1762) and is given in a more convenient form by Ališauskas (1983) as (35*b*). The Wigner coefficient (3.5) differs from the isofactor by a simple factor $(\dim_{SO_n}(l_3))^{-1/2}$. The phases of both (3.2) and (3.5) are changed in comparison with Ališauskas (1983)†. The expression (3.5) may be replaced by the $3jm$ symbol (Wigner coefficient) of SU_2

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & -m_3 \end{pmatrix} \tag{3.7a}$$

for $n=3$ and by

$$\begin{pmatrix} \frac{1}{2}l_1 & \frac{1}{2}l_2 & \frac{1}{2}l_3 \\ \frac{1}{2}m_1 & \frac{1}{2}m_2 & -\frac{1}{2}m_3 \end{pmatrix}^2 \tag{3.7b}$$

for $n=4$.

The symmetry properties of the Clebsch-Gordan and Wigner coefficients allow us to contract the domain of the summation parameters. The factor $(2-\delta_{m_1 0}\delta_{m_2 0})$ should be included in the RHS of (3.1), if the parameter m_1 is restricted to non-negative values and for $m_1=0$, $m_2 \geqslant 0$.

In order to obtain the particular case of (3.1) with $h_3=0$, the special Clebsch-Gordan coefficients of the chain $U_n \supset (U_2 \dotplus U_{n-2}) \supset SO_2 \dotplus SO_{n-2}$ were transformed similarly as was done for the isofactors of the $SU_3 \supset SO_3$ Elliott states by Engeland (1965) (see Vergados 1968, Asherova and Smirnov 1970). The absolute value of the special isofactor

† The changes of phase correspond to the transposition of the chains of subgroups $U_2 \supset SO_2$ and $U_{n-2} \supset SO_{n-2}$ of U_n.

of $U_n \supset U_2 \dot{+} U_{n-2}$ is equal to 1 in this case because all the states of the subgroup U_{n-2} are chosen to be scalar and the Young tableaux of the corresponding irreps of U_n and U_2 coincide. The normalisation and phase factors for this particular case of the isofactor (3.1) are found from the comparison with particular cases of (A4.2) of Ališauskas (1984).

The most general case of (3.1) is obtained in the following way. Let us expand E_{32}^{j+m} and E_{31}^{j-m} on the RHS of (2.9) as follows (cf Ališauskas and Norvaišas 1980):

$$E_{3i}^{a} = a! \sum_{l_i' \mu_i} \left| \begin{matrix} [a]_n \\ l_i' \\ \mu_i \end{matrix} \right\rangle_{(3)} \left\langle \begin{matrix} [a]_n \\ l_i' \\ \mu_i \end{matrix} \right|_{(i)} \qquad (i=1,2) \tag{3.8}$$

where μ_i are the basis labels of the irrep l_i' of SO_n included in U_n. The direct product of polynomials depending on η_{3s} (bra states) may now be expanded in terms of the states of the irrep $[p_3]$ of U_n. The Clebsch-Gordan coefficients of $U_n \supset SO_n \supset \ldots$ which appeared are to be used as the coupling coefficients for the ket states. In this way the ket states for the symmetric irrep $[2j, 0]$ of both complementary groups U_2 and U_n are obtained, the action of which into the bra state in the RHS of (2.9) may be expanded in terms of the direct product states depending on η_{1s} and η_{2s}. For this purpose the above-mentioned particular case of (2.9) and the elementary isofactors of $U_{2n} \supset U_2 \times U_n$ (expressed in terms of the dimensions of irreps of the complementary symmetric groups S_N (cf Vanagas 1971 (13.25))) are useful as well as the Clebsch-Gordan coefficients of SU_2 and the simple reduced matrix elements. The use of the symmetry and orthogonality properties of SU_2 Clebsch-Gordan coefficients allows us to represent the expansion coefficient of the special coupled states of $U_n \supset SO_n$ in terms of the direct product states (coupled in frames of SO_n) as (3.1).

The following expression for the special isofactor is of importance, especially in § 4:

$$\begin{aligned} &\begin{bmatrix} [h_1'h_2']_n & [p_3 0]_n & [h_1h_2h_3]_n \\ 0 & 0 & 0 \end{bmatrix} \\ &\quad = \left(\frac{(n-2)(h_1-h_2+1)(h_2-h_3+1)(h_1-h_3+2)p_3!!(h_1+n-2)!!}{(p_3+n-2)!!(h_1+2)!!(h_2+1)!!h_3!!(h_1'+n-2)!!(h_2'+n-3)!!} \right. \\ &\quad \times \frac{(h_2+n-3)!!(h_3+n-4)!!(h_1'+1)!!h_2'!!(h_1-h_1'-1)!!(h_2-h_2'-1)!!}{(h_1-h_1')!!(h_2-h_2')!!(h_1-h_2'+1)!!} \\ &\quad \times \left. \frac{(h_1-h_2')!!(h_1'-h_2-1)!!(h_2'-h_3-1)!!(h_1'-h_3)!!}{(h_1'-h_2)!!(h_2'-h_3)!!(h_1'-h_3+1)!!} \right)^{1/2}. \end{aligned} \tag{3.9}$$

The particular cases of (3.9) with $h_3 = 0$ and with $h_1' = h_2$ follow from (3.1), the first immediately and the second after use of the Saalschutz summation theorem (see Slater 1966, § 2.3). The third particular case with $h_1 = h_1'$, $h_2 = h_2'$, $h_3 = p_3$ is the inverse of the coefficient before

$$\left(\frac{(n-2)!!}{h_3!!(h_3+n-2)!!} \right)^{1/2} W_{33}^{h_3/2} \left| \begin{matrix} [h_1h_2]_n \\ (\text{HW}); 0 \end{matrix} \right\rangle \tag{3.10}$$

in the expansion of HWS according to (2.9)†. The proof of the general case of (3.9) will be discussed in the appendix.

† The unnormalised HWS of (2.9) type was in fact found by projecting from the direct product state (3.10) (see the reasoning about overlaps for the states of two parametric irreps by Ališauskas (1984)).

4. Another expression for the special isofactors

The expression (3.1) simplifies considerably for low values of h_3. The alternative expression may be more convenient for other values of the parameters. To obtain this expression, we use the expansion of the $SU_n \supset SO_n$ direct product states in terms of the coupled states similarly as was done for the special $SU_3 \supset SO_3$ isofactors by Hecht and Suzuki (1983). The normalised direct product state coupled in frames of SO_n may be chosen in the form

$$\left|\begin{matrix}[p_1]_n \times [p_2]_n \times [p_3]_n \\ \bar{l}_1 \times \bar{l}_2 \times l_3; 0\end{matrix}\right\rangle = \left(\frac{(n-3)!!\, l_3!\,(2\bar{l}_1+n-2)!!\,(2\bar{l}_2+n-2)!!}{(2l_3+n-4)!!\,(l_3+n-3)!}\right)^{1/2} \frac{W_n(p_3 l_3)}{W_n(p_1\bar{l}_1)\, W_n(p_2\bar{l}_2)}$$
$$\times \sum_{x,m} \frac{(-1)^x 2^{-x}(2l_3-2x+n-4)!!}{x!\,(p_3+l_3-2x)!!\,(p_3+l_3-2x+n-2)!!}$$
$$\times \left(\frac{(2x)!!\,(2x+n-2)!!}{(l_3-2x)!\,(\bar{l}_1-x-m)!\,(\bar{l}_2-x+m)!}\right)^{1/2}$$
$$\times \begin{bmatrix} x & \frac{1}{2}l_3-x & \frac{1}{2}l_3 \\ m & \frac{1}{2}(\bar{l}_1-\bar{l}_2)-m & \frac{1}{2}(\bar{l}_1-\bar{l}_2)\end{bmatrix} E_{13}^{\bar{l}_1-x-m} E_{23}^{\bar{l}_2-x+m} W_{33}^{(p_3+l_3)/2-x}$$
$$\times W_{11}^{(p_1-\bar{l}_1)/2} W_{22}^{(p_2-\bar{l}_2)/2} \left|\begin{matrix}[2x, 0]_n \\ x, x, m; 0\end{matrix}\right\rangle. \tag{4.1}$$

Here $\bar{l}_1+\bar{l}_2=l_3$. In order to prove (4.1) one needs to use the relation by Asherova *et al* (1980), given as (2.5) in Ališauskas (1984), and expand the RHS of (4.1) in terms of W_{ij} (including W_{13} and W_{23}). An identical result may be obtained by acting on

$$\left(\frac{(n-1)!}{\bar{l}_1!\,\bar{l}_2!\,(l_3+n-1)!}\right)^{1/2} W_{13}^{\bar{l}_1} W_{23}^{\bar{l}_2} \tag{4.2}$$

with the projection operator of the complementary group Sp(2, R),

$$P_{p_3,l_3}^{l_3(n)} = \frac{1}{W_n(p_3 l_3)} \left(\frac{(2l_3+n-2)}{(2l_3+n-4)!!}\right)^{1/2}$$
$$\times \sum_x \frac{(-1)^x}{(2x)!!}(2l_3-2x+n-4)!!\, W_{33}^{(p_3-l_3)/2+x}\, \bar{W}_{33}^{x}, \tag{4.3}$$

after multiplication with the simple factor

$$\left(\frac{\dim_{SU_n}(l_3)}{\dim_{SO_n}(l_3)}\right)^{1/2} = \left(\frac{(l_3+n-1)(l_3+n-2)}{(n-1)(2l_3+n-2)}\right)^{1/2} \tag{4.4}$$

which is the inverse of some isofactor of $U_n \supset SO_n$.

The projection operator (4.3) leaves in the function (4.2) only the polynomials in η_{3s} transforming according to the irrep l_3 of SO_n. It is evident that the polynomials in η_{1s} and η_{2s} belong to irreps $\bar{l}_1$ and $\bar{l}_2$, respectively. However, in the case $l_1+l_2>l_3$ it is necessary to use analogous projection operators depending on $\bar{W}_{11}$, W_{11} and W_{22}, $\bar{W}_{22}$, and the direct product state obtained is too bulky to be useful as a generating function.

Now the RHS of (4.1) may be expanded in terms of the $U_3\times U_n$ states as defined in § 2. The special isofactors of (3.9) type, the SU_2-tensorial properties and reduced matrix elements of the operators, as well as the recoupling technique, were used and the following expression for the special isofactors was obtained (for $\bar{l}_1+\bar{l}_2=l_3$):

$$\sum_{\omega}\begin{bmatrix}[p_1 0] & [p_2 0] & [\alpha+I,\alpha-I]_n\\ \bar{l}_1 & \bar{l}_2 & \omega l_3\end{bmatrix}\begin{bmatrix}[\alpha+I,\alpha-I] & [p_3 0] & [h_1h_2h_3]_n\\ \omega l_3 & l_3 & 0\end{bmatrix}$$

$$=(-1)^{\delta_{n,3}p_3}\frac{W_n(p_3l_3)(p_1-\bar{l}_1)!!(p_2-\bar{l}_2)!!\times[(h_1-h_2+1)(h_2-h_3+1)(h_1-h_3+2)]^{1/2}}{W_n(p_1,\bar{l}_1)W_n(p_2,\bar{l}_2)S_{3,2}[h_1h_2h_3;\alpha+I,\alpha-I]}$$

$$\times\left(\frac{(h_1+n-2)!!(h_2+n-3)!!(h_3+n-4)!!\times(2\bar{l}_1+n-2)!!(2\bar{l}_2+n-2)!!}{(n-4)!!(2l_3+n-4)!!(l_3+n-3)!(h_1+2)!!(h_2+1)!!h_3!!}\right)^{1/2}$$

$$\times\sum_{x,j_1,j_2}\frac{(-1)^x(2j_1+1)(2j_2+1)(2l_3-2x+n-4)!!(j_1+j_2-x-1)!!}{(\alpha-\frac{1}{2}l_3-j_1)!!(\alpha-\frac{1}{2}l_3+j_1+1)!!(x-j_1+j_2)!!(x+j_1-j_2)!!}$$

$$\times\frac{(\alpha-\frac{1}{2}l_3+x-j_2)!!(\alpha-\frac{1}{2}l_3+x+j_2+1)!!\nabla(\frac{1}{2}l_3,j_1,I)}{(x+j_1+j_2+1)!!(p_3+l_3-2x+n-2)!!\nabla^2(\frac{1}{2}l_3-x,j_2,I)}$$

$$\times\left(\frac{(j_1+m_1-1)!!(j_1-m_1-1)!!}{(j_1+m_1)!!(j_1-m_1)!!}\right)^{1/2}\begin{bmatrix}j_1 & \frac{1}{2}l_3 & I\\ m_1 & \frac{1}{2}(\bar{l}_1-\bar{l}_2) & \frac{1}{2}(p_1-p_2)\end{bmatrix}$$

$$\times\frac{(h_1+\frac{1}{2}l_3-\alpha-x-j_2-1)!!(h_1+\frac{1}{2}l_3-\alpha-x+j_2)!!\times(h_2+\frac{1}{2}l_3-\alpha-x+j_2-1)!!}{(\alpha-\frac{1}{2}l_3-h_2+x+j_2)!!(\alpha-\frac{1}{2}l_3-h_3+x-j_2)!!\times(\alpha-\frac{1}{2}l_3-h_3+x+j_2+1)!!}. \qquad (4.5)$$

Here $m_1=\frac{1}{2}(p_1-p_2-\bar{l}_1+\bar{l}_2)$, j_1, j_2, x are integers and $\alpha-\frac{1}{2}l_3-j_1$, $x-j_1+j_2$, $x+j_1-j_2$ and $j_1\pm m_1$ are even integers.

The concept of dual bases (Ališauskas 1978, 1984) allows us to use (4.5) (with $p_1=\alpha+I$, $p_2=\alpha-I$) as the special coupling coefficients for the quasi-stretched basis states of the irrep $(\lambda\nu\dot{0})$ as defined by Ališauskas (1984) and the basis states of the symmetric irrep $[p_3]$ of U_n. The first isofactors in the left-hand side of (4.5) may be found from the corrected equation (7.4) of Ališauskas (1984) (in this case $\lambda=2I$, $\nu=\alpha-I$, $L_1=l_3$, $L_2=0$) and in such a way the complete system of equations for the second isofactors may be obtained.

Otherwise the bilinear combination of isofactors defined by the left-hand side of (3.1) may be expressed in terms of the above-mentioned particular case of (4.5) and the corrected equation (6.9) of Ališauskas (1984). The product of both quantities should be summed over values of the multiplicity labels restricted by the inequality (6.13) of Ališauskas (1984). One can abandon this condition if the pseudo-isofactors are used instead of the proper isofactors (see § 6 of Ališauskas (1984)). In this way the expression for the left-hand side of (3.1) may be obtained which has seven sums.

5. Relation with the isofactor for the closed shells case

The isofactors represented by equation (3.1) or (4.5) are in fact resubduction coefficients between the chains of the complementary groups

$$\begin{array}{ccc} \mathrm{Sp}(6, R) \supset \mathrm{Sp}(4, R) & \dot{+} & \mathrm{Sp}(2, R) \\ \cup & & \cup \\ \mathrm{Sp}(2, R) \dot{+} \mathrm{Sp}(2, R) & & \mathrm{U}_1 \\ \cup \qquad\quad \cup & & \\ \mathrm{U}_1 \qquad\quad \mathrm{U}_1 & & \end{array} \tag{5.1a}$$

and

$$\begin{array}{c} \mathrm{Sp}(6, R) \supset \mathrm{U}_3 \supset \mathrm{U}_2 \supset \mathrm{U}_1 \\ \cup \\ \mathrm{U}_1 \dot{+} \mathrm{U}_1' \end{array} \tag{5.1b}$$

for the irrep $\langle n/2, n/2, n/2\rangle$ of Sp(6, R). Clearly, the states of the same irrep of Sp(6, R) may be realised as states of the irrep $[fff]$ of the complementary group O_{n-2f}. Therefore the following dependence exists between the special isofactors for the groups $SU_n \supset SO_n$ and $SU_n \supset SO_n$ $(n' = n+2f)$ (cf Ališauskas 1983):

$$\sum_{\omega} \begin{bmatrix} [p_1 0] & [p_2 0] & [g_1, g_2]_n \\ l_1 & l_2 & \omega[l_3 f] \end{bmatrix} \begin{bmatrix} [g_1 g_2] & [p_3 0] & [h_1 h_2 h_3]_n \\ \omega[l_3 f] & l_3 & [fff] \end{bmatrix}$$
$$= \sum_{\omega'} \begin{bmatrix} [p_1 - f0] & [p_2 - f0] & [g_1 - f, g_2 - f]_{n+2f} \\ l_1 - f & l_2 - f & \omega', l_3 - f \end{bmatrix}$$
$$\times \begin{bmatrix} [g_1 - f, g_2 - f, 0] & [p_3 - f0] & [h_1 - f, h_2 - f, h_3 - f]_{n+2f} \\ \omega', l_3 - f & l_3 - f & 0 \end{bmatrix}. \tag{5.2}$$

Here $h_1 - f$, $h_2 - f$, $h_3 - f$ are even integers, as well as $p_1 - l_1$, $p_2 - l_2$, $p_3 - l_3$.

6. Concluding remarks

We obtained two classes of expression for the isoscalar factors by means of dual approaches, similar to Hecht and Suzuki (1983). If we pass over the difference in the complexity of the problems, a distinguishing feature of our method is the generating functions in terms of the creation and annihilation operators, coupled to the elementary scalar generators of Sp(6, R) and its compact subgroup, when exclusively creation operators were used in a similar situation by Hecht and Suzuki (1983). The generating functions in terms of solely (exclusively) creation operators are quite natural for coupling to the mixed tensor of U_n (cf Ališauskas 1984), for example, $[p_1 0] \times [p_2 0] \times [\dot{0}, -q]$ to $[h_1, h_2, \dot{0}, -h_n]$. The corresponding situation may be described in the framework of the representation theory of the complementary groups $\mathrm{Sp}(4, 2) \supset \mathrm{U}(2, 1)$. Otherwise, the corresponding isofactors may also be obtained from those under consideration by means of the substitution group technique (Ališauskas and Jucys 1967b, see Ališauskas 1983, 1984).

In reality equations analogous to (2.9), (3.1), (4.1) and (4.5) corresponding to the mixed tensor case were obtained earlier as (2.9), (3.1), (4.1) and (4.5) themselves. The substitution group technique allowed us to make predictions about the structure of (2.9) and (4.1) for the covariant tensor case. Since we do not know any application of the special isofactors of such type which couple to the mixed tensors, this scaffold was removed as much as possible and direct and shorter proofs were found for (2.9) and (3.1) but not for (4.1). In this last case attempts to anticipate and to prove (4.1) without use of the generating function applicable for the mixed tensor case were not successful, perhaps because of non-commutativity of the different operators in the RHS of (4.1).

Although in this multiplicity-free case two classes of expression for the isofactors are not essentially dual (numerically they are equivalent), the method of § 4 was generalised for consideration of the stretched states of two-parametric irreps of $SU_n \supset SO_n$ and the corresponding isofactors. A class of effective analytical relations between biorthogonal systems of isofactors (respectively, resubduction coefficients or recoupling matrices) induced by this investigation will be presented in future papers.

Appendix. On the proof of (3.9)

In order to prove the expression (3.9) for the special isofactors of $U_n \supset SO_n$ we use the recurrence relation

$$\begin{aligned}
&\begin{bmatrix} [p_3-2] & [2] & [p_3]_n \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} [h_1'h_2'] & [p_3] & [h_1h_2h_3]_n \\ 0 & 0 & 0 \end{bmatrix} \\
&\quad = \sum_{[\lambda]_n} \langle [h_1'h_2']; [p_3-2][2]([p_3])[h_1h_2h_3]| \\
&\quad \times |[h_1'h_2'][p_3-2]([\lambda]_n)[2][h_1h_2h_3]\rangle \\
&\quad \times \begin{bmatrix} [h_1'h_2'] & [p_3-2] & [\lambda]_n \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} [\lambda]_n & [2] & [h_1h_2h_3]_n \\ 0 & 0 & 0 \end{bmatrix}.
\end{aligned} \tag{A1}$$

In the RHS of (A1) the recoupling matrix of U_n (equivalent to one of U_3) appeared with Young tableaux $[\lambda]_n$ taking the values $[h_1, h_2, h_3-2]$, $[h_1, h_2-2, h_3]$ and $[h_1-2, h_2, h_3]$. This recoupling matrix may be found from (1) and (B1) of Ališauskas *et al* (1972). The special isofactor of $U_n \supset SO_n$ in the LHS of (A1) is a particular case of (A4.7) of Ališauskas (1984). The particular cases of (A1) and (3.9) with $h_1 = h_1'$, $h_2 = h_2'$, $p_3 = h_3$ allowed us to find the special isofactor

$$\begin{bmatrix} [h_1h_2h_3-2]_n & [2] & [h_1h_2h_3]_n \\ 0 & 0 & 0 \end{bmatrix} = \left(\frac{(h_3+n-4)(h_1-h_3+3)(h_2-h_3+2)}{n(h_3-1)(h_1-h_3+4)(h_2-h_3+3)}\right)^{1/2}. \tag{A2}$$

The special elements of the substitution group (Ališauskas and Jucys 1967b) lead to expressions

$$\begin{bmatrix} [h_1, h_2-2, h_3]_n & [2] & [h_1h_2h_3]_n \\ 0 & 0 & 0 \end{bmatrix} = \left(\frac{(h_2+n-3)(h_1-h_2+2)(h_2-h_3)}{nh_2(h_1-h_2+3)(h_2-h_3-1)}\right)^{1/2}, \tag{A3}$$

$$\begin{bmatrix} [h_1-2, h_2, h_3]_n & [2] & [h_1h_2h_3]_n \\ 0 & 0 & 0 \end{bmatrix} = \left(\frac{(h_1+n-2)(h_1-h_2)(h_1-h_3+1)}{n(h_1+1)(h_1-h_2-1)(h_1-h_3)}\right)^{1/2}. \tag{A4}$$

Now it is not difficult to write (A1) explicitly and to check that (3.9) satisfies (A1).

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